A Statistical Learning View of Simple Kriging

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Introduction
Geostatistical context

Geo-databases: information from **location and value** of the data

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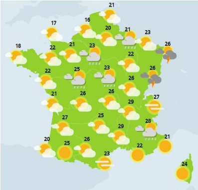

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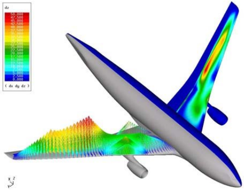

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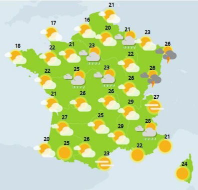

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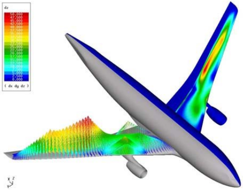

Different from usual statistical learning theory:
non independent observations

Machine Learning:

Spatial Analysis:

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- **Limits:** lack of nonasymptotic results

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Objective: develop framework for **kriging** based on (nonasymptotic) study of the performances of a (nonparametric) **covariance estimator**

Preliminaries

1. Simple Kriging
2. Kernel Ridge Regression

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- $\Sigma(\mathbf{s}_n) = Var(\mathbf{X}(\mathbf{s}_n)) = (Cov(X_{s_j}, X_{s_i}))_{1 \le i,j \le n}$ covariance matrix

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  $\longrightarrow$ $\checkmark$ allows to **control the spectrum** of the covariance matrix

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Simple Kriging: **predict** the value of X at some unobserved location s, based on n sampled observations $(X_{s_i})_{i \leq n}$, assuming a **linear combination** of the observations: $f_\lambda(s) = \langle \lambda(s), \mathbf{X}(\mathbf{s}_n) \rangle$.

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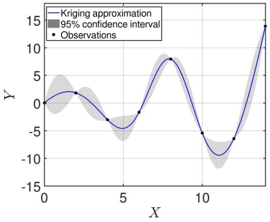
Remark (Exact Interpolation):

- at observed points: exact interpolator
- at unobserved points: linear combination of data



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$\Longrightarrow$ need to establish **rate bounds** that assess the **generalization capacity** of the resulting predictive map

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$$\Longrightarrow \qquad \hat{f}(x) = {}^t\, \mathbf{Y}_n \left(\eta I_n + \mathscr{K}_n \right)^{-1} \kappa_n(x)$$

Gram matrix $\longleftrightarrow$ $\longleftrightarrow$ kernel vector

Main Results

1. Connections between SK and KRR
2. Nonasymptotic bound for the Excess of Risk
3. Illustrative Numerical Experiments

Main Results

Connections: ML ⟷ Geostat

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- **goal:** build predictor for new unobserved data

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$\Longrightarrow$ kriging predictor **same** form as kernel ridge regressor:

- regularised Gram matrix ⟷ covariance matrix
- kernel vector ⟷ covariance vector

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  $\longrightarrow$ $\checkmark$ taken into account when studying the **estimation error** of the covariance function for **all lags** (**smoothness** of the covariance function)

Main Results

Nonasymptotic bound for Excess of Risk

Theorem ([Siviero et al., 2022])

For any $\delta \in (0, 1)$, we have with probability at least $1 - \delta$:

$$L_S(f_{\widehat{\lambda}}) - L_S(f_{\lambda^*}) \leq C_6 \frac{\log(n/\delta)}{n} + \frac{2D^2}{n+1},$$

as soon as $n \geq C_6' \log(n/\delta)$, where C_6 and C_6' are nonnegative constants depending on j_1, m and M solely.

Sketch of Proof

Poisson tail bounds

Hyp 5
bounded eigenvalues

Poisson tail bounds for deviations $|\widehat{\gamma}(h) - \gamma(h)|$ and $|\widehat{c}_h(0) - c(0)|$

↑ ↑

Hyp 2
Gaussian hyp

Distributions of estimators $\widehat{c}_h(0)$ and $\widehat{\gamma}(h)$

Inferior bound on $\mathbf{n_h}$

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Bound on covariance estimation error

Hyp 6
regularity hyp

Bound on covariance estimation error for **all lags**

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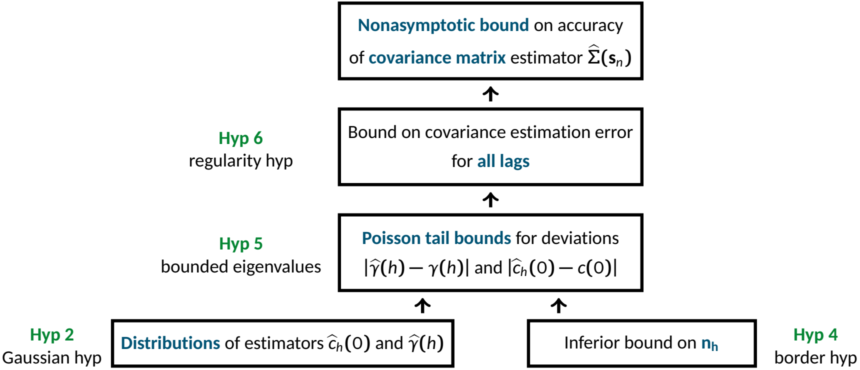

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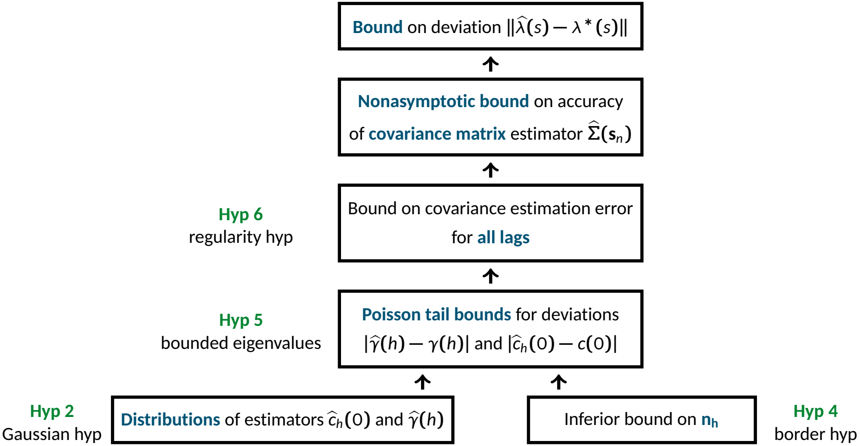
$\Longrightarrow$ nonasymptotic bound on accuracy of covariance matrix estimator $\rho\left(\widehat{\boldsymbol{\Sigma}}(\mathbf{s}_n) - \boldsymbol{\Sigma}(\mathbf{s}_n)\right)$

Sketch of Proof

$$\forall s \in S, \quad \left(f_{\widehat{\lambda}}(s) - f_{\lambda^*}(s)\right)^2 \leq \|\widehat{\lambda}(s) - \lambda^*(s)\|^2 \times \|\mathbf{X}(\mathbf{s}_n)\|^2$$



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Maps of MSE on 100 realisations, with two covariance models (for fixed j_1)

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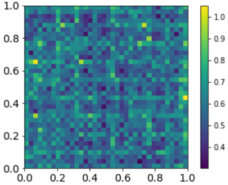

Figure 1: *Truncated power law* (TPL)

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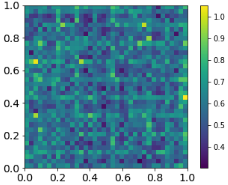

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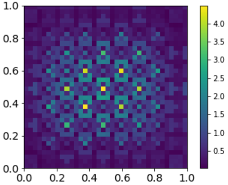

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✗ not **Hyp 4**, but vanishes quickly

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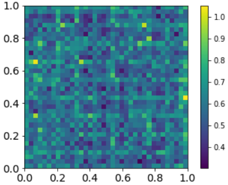

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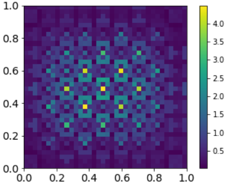

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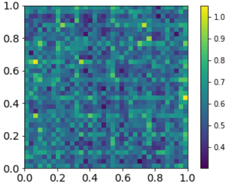

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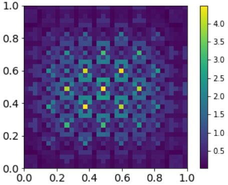

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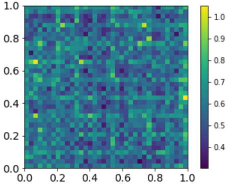

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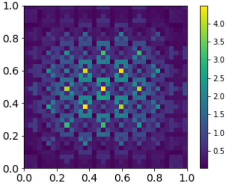

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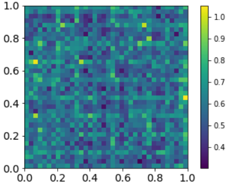

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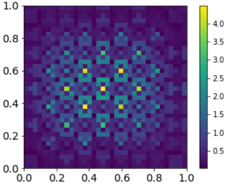

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$\Longrightarrow$ results for Gaussian covariance encourages to relax **Hyp 4**

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- **partitioning methods** (dyadic CART): divide spatial domain into clusters where the process is *locally stationary*

Conclusion

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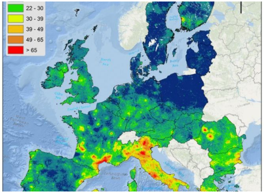

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- novel theoretical framework offering **guarantees** for empirical simple Kriging rules in the form of **non-asymptotic bounds**
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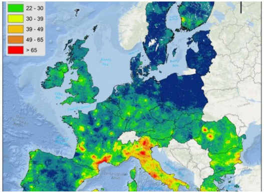
- **possible extensions** to a more general framework



Thanks for your attention !

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